4 The Story of Numbers

Number System *A well-defined writing system for expressing numbers using digits or other symbols in a consistent manner, is called a number system.*

Essential Requirements in a Number System Ideally, a number system should fulfil the following requirements:

1. It should contain a set of symbols and digits so as to give every written number a unique representation.
2. It should support or enable an unending standard sequence of numbers that is easy to count with.
3. It should be such that every possible number can be expressed using the defined symbols or digits. There should not be a limitation to the size of the numbers that can be expressed in that system nor should there be a need to devise more symbols to represent larger numbers.

Here, you will agree that the Hindu numeral system that we use today, comprising digits 0 to 9, fulfils all the above criteria and hence has been well accepted internationally as a standard number system.

Early Counting

As early humans settled down in civilisations during the Stone Age, they felt the need for counting for three basic purposes.

(i) To keep track of prey animals, livestock or stored goods.

(ii) To count members of a tribe living at one place or a smaller group while shifting from one place to another.

(iii) To track time by following up with the cycles of the moon or the change of seasons.

Some of the earliest methods of counting were:

(i) by using fingers or other body parts;

(ii) by using objects like pebbles, sticks or any object available in abundance;

(iii) by making marks or notches on bones or cave walls—an early, crude adaptation of the system of tally marks that we use today in statistics and data keeping.

However, they had their own symbols or numbers and they did not make use of the numbers that we use today.

Evolution of the Hindu Numeral System

The structure of the modern oral and written numbers that we use today originated about 3000 years ago during the Vedic period. Ancient Indian texts such as the *Yajurveda Samhita* mentioned names of numbers based on powers of 10, going up to 10^{12} (i.e., one trillion).

Present day notation	Name in Yajurveda Samhita
1	*eka*
$10\ (= 10^1)$	*dasha*
$100\ (= 10^2)$	*shatha*
$1000\ (= 10^3)$	*sahasra*
$10000\ (= 10^4)$	*ayuta*
10^5	*niyuta*
10^6	*prayuta*

Present day notation	Name in Yajurveda Samhita
10^7	*arbuda*
10^8	*nyarbuda*
10^9	*samudra*
10^{10}	*madhya*
10^{11}	*anta*
10^{12}	*parardha*

The modern number system based on the digits 0–9 was developed in India around 2000 years ago. An early example of this system is found in the *Bakshali* manuscript written around 3rd century CE. This system was a place-value system with a dot to denote the zero, called the *shunyasthana*.

Aryabhata was the first mathematician to fully explain and elaborate scientific computations with the Indian system of 10 symbols around 499 CE.

In 628 CE, astronomer-mathematician Brahmagupta wrote *Brahma-sphuta-siddhanta* which defined zero (*shunya*) as the result of subtracting a number from itself and postulated negative numbers. By the end of 7th century CE decimal numbers began to appear in inscriptions in South-east Asia as well as in India.

Spread to Arab World and Europe

The Indian number system was transmitted to the Arab world by around 800 CE. It gained popularity with its use in the work of the Persian mathematician Al-Khwarizmi titled 'On the Calculation with Hindu Numerals' and the Arab mathematician Al-Kindi who wrote four volumes titled 'On the Use of the Indian Numerals'.

With the translation of Al-Khwarizmi's work into Latin the Hindu numerals were transmitted to Europe. But Fibonacci, an Italian mathematician, is said to have promoted the Indian numerals in Europe with his book '*Liber Abaci*' published in 1202 CE. However, the Roman numerals were so popular in Europe at that time that the Indian numerals did not come into wide use until the invention of printing and the European Renaissance.

However, with the adoption of Indian numerals in Europe, their use spread to all the continents around the world.

Nomenclature

(i) Arab scholars like Al-Khwarizmi and Al-Kindi called the Indian numerals '*Al Hind*' meaning 'from India'.

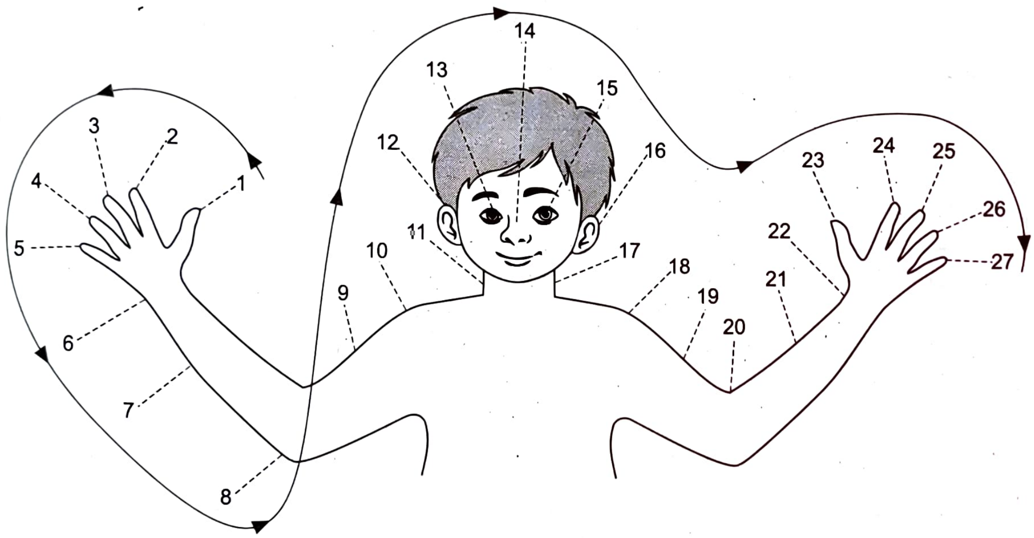
(ii) European scholars called them 'Arabic numerals' as they had learned them from the Arab world.

(iii) As the use of Indian numerals spread to every continent and to every corner of the world, the recent textbooks and documents around the world renamed them as 'Hindu Numerals' or 'Hindu-Arabic Numerals'. Here the word 'Hindu' denoted 'a geography or people from whom these numbers came' and not a religion.

Some Early Number Systems

I. Use of Body Parts

Many groups of people across the world have devised and used number systems using body parts for counting. Such systems use different points on the body to represent numbers, extending beyond the fingers. One such system is the 27-part system used by the Oksapmin people of Papua New Guinea. They begin counting from the thumb of one hand, followed by the fingers in order and move up the arm, neck and head to a halfway point touching specific identified points. They then proceed down the other side of the body in a symmetrical pattern, ending up on the little finger of the other hand, as shown in the diagram below.



II. Tally Marks on Bones and Other Surfaces

Yet another primitive method of counting was by using tally marks. The early tally marks during the prehistoric times were rather crude notches, i.e., marks etched on a surface such as a bone or a wall of a cave. In this method, a mark is made for each object being counted. And the final total number of marks drawn represents the total number of objects.

Archaeologists have unearthed such notched bones dating back more than 20000 years. The oldest known such bones are:

(i) **The Ishango Bone** Found in the Democratic Republic of Congo and dated more than 20000 years ago, it had notches arranged in three columns, possibly indicating calendrical systems.

(ii) **The Lebombo Bone** Discovered in South Africa, this baboon bone is about 44000 years old and has 29 distinct notches, which possibly served as a tally stick or a lunar calendar.

III. Number Names Obtained by Counting in Twos

(i) **Gumulgal System** Gumulgal refers to an indigenous group of people in Australia who are known for their number system, which was based on counting in twos. It was clearly an additive system using the symbol '*urapon*' for one and '*ukasar*' for two. Thus, they wrote number names as shown below.

Number	Name in Gumulgal system	Meaning
1	*urapon*	1
2	*ukasar*	2
3	*ukasar-urapon*	2 + 1
4	*ukasar-ukasar*	2 + 2
5	*ukasar-ukasar-urapon*	2 + 2 + 1
6	*ukasar-ukasar-ukasar*	2 + 2 + 2

Any number greater than 6 was termed '*ras*'. Gumulgal system is considered a significant landmark in the history of numbers because it introduced the idea of counting in a certain group size and using it to represent numbers. This idea laid the foundation of complex number systems that came up later.

(ii) Bushmen of South Africa The Bushmen of South Africa too devised a number system based on using the following symbols.

Symbol	*xa*	*t'oa*	*'quo*	*t'a*
Meaning	1	2	3	increment of 1, i.e., +1

Thus, they represented numbers as shown below.

Number	Representation	Meaning
1	*xa*	1
2	*t'oa*	2
3	*'quo*	3
4	*t'oa-t'oa*	2 + 2
5	*t'oa-t'oa-t'a*	2 + 2 + 1
6	*t'oa-t'oa-t'oa*	2 + 2 + 2

(iii) Bakairi Number System The Bakairis (a group of indigenous people) of South America used an additive system where numbers are formed by combining words for one (*tokale*) and two (*ahage*). Thus, they represented numbers as shown below.

Number	Representation	Meaning
1	*tokale*	1
2	*ahage*	2
3	*ahage-tokale*	2 + 1
4	*ahage-ahage*	2 + 2
5	*ahage-ahage-tokale*	2 + 2 + 1
6	*ahage-ahage-ahage*	2 + 2 + 2

Example 1 *Consider the extension of the Gumulgal number system beyond 6 in the same way of counting by 2s. Come up with ways of performing the different arithmetic operations (+, −, ×, ÷) for numbers occurring in this system, without using Hindu numerals. Use this to evaluate the following:*

(i) (ukasar-ukasar-ukasar-ukasar-urapon) + (ukasar-ukasar-ukasar-urapon)

(ii) (ukasar-ukasar-ukasar-ukasar-urapon) − (ukasar-ukasar-ukasar)

(iii) (ukasar-ukasar-ukasar-ukasar-urapon) × (ukasar-ukasar)

(iv) (ukasar-ukasar-ukasar-ukasar-ukasar-ukasar-ukasar-ukasar) ÷ (ukasar-ukasar)

SOLUTION For ease in calculations, we may use the symbols 'uk' for *ukasar* and 'ur' for *urapon*.

(i) Required sum

= (uk-uk-uk-uk-ur) + (uk-uk-uk-ur)
= uk + uk + uk + uk + ur + uk + uk + uk + ur
= uk + uk + uk + uk + uk + uk + uk + ur + ur
= uk + uk + uk + uk + uk + uk + uk + uk [∵ ur + ur = uk]
= uk-uk-uk-uk-uk-uk-uk-uk. [∵ uk + uk = uk-uk]

Hence, required sum

= *ukasar-ukasar-ukasar-ukasar-ukasar-ukasar-ukasar-ukasar*

(ii) Required difference

= (uk-uk-uk-uk-ur) − (uk-uk-uk)
= (uk + uk + uk + uk + ur) − (uk + uk + uk)
= uk + uk + uk + uk + ur − uk − uk − uk
= uk + ur = uk-ur.

Hence, required difference = *ukasar-urapon*.

(iii) Required product

= (uk-uk-uk-uk-ur) × (uk-uk)
= (uk + uk + uk + uk + ur) × (uk + uk)
= uk × (uk + uk + uk + uk + ur) + uk × (uk + uk + uk + uk + ur)
[by distributive law]
= uk × uk + uk × uk + uk × uk + uk × uk + uk × ur + uk × uk + uk × uk + uk × uk + uk × uk + uk × ur
= uk-uk + uk-uk + uk-uk + uk-uk + uk + uk-uk + uk-uk + uk-uk + uk-uk + uk
[∵ uk × uk = uk-uk and uk × ur = uk]
= uk-uk-uk-uk-uk-uk-uk-uk-uk-uk-uk-uk-uk-uk-uk-uk-uk-uk.

Hence, required product

= *ukasar-ukasar-ukasar* ... 18 *times*.

(iv) Required quotient

= (uk-uk-uk-uk-uk-uk-uk-uk) ÷ (uk-uk)

$$= \frac{(\text{uk} + \text{uk} + \text{uk} + \text{uk} + \text{uk} + \text{uk} + \text{uk} + \text{uk})}{(\text{uk} + \text{uk})}$$

$$= \frac{(\text{uk} + \text{uk})}{(\text{uk} + \text{uk})} + \frac{(\text{uk} + \text{uk})}{(\text{uk} + \text{uk})} + \frac{(\text{uk} + \text{uk})}{(\text{uk} + \text{uk})} + \frac{(\text{uk} + \text{uk})}{(\text{uk} + \text{uk})}$$

= ur + ur + ur + ur
= uk + uk [∵ ur + ur = uk]
= uk-uk.

Hence, required quotient = *ukasar-ukasar*.

IV. The Roman Numerals

As the name suggests, this numeral system originated in ancient Rome and remained in prevalence throughout Europe for centuries. In this system, numbers are written with combinations of letters from the Latin alphabet, each denoting a fixed integer value.

The landmark numbers along with their basic symbols, used in Roman numeral system, are as shown below.

Symbol	I	V	X	L	C	D	M
Landmark number	1	5	10	50	100	500	1000

If a bar is placed over a numeral, it is multiplied by 1000.

Thus, $\overline{V} = 5000$ and $\overline{X} = 10000$, etc.

Using these symbols, we may form all Roman numerals by adopting the rules given below.

RULE 1 *Repetition of a symbol in a Roman numeral means addition.*

CAUTIONS (i) Only I, X, C, M can be repeated.

(ii) V, L and D are never repeated.

(iii) No symbol in a Roman numeral can be repeated more than 3 times.

EXAMPLES (i) II = (1 + 1) = 2. (ii) XX = (10 + 10) = 20.

(iii) XXX = (10 + 10 + 10) = 30. (iv) CC = (100 + 100) = 200.

RULE 2 *A smaller numeral written to the right of a larger numeral is always added to the larger numeral.*

EXAMPLES (i) VI = (5 + 1) = 6. (ii) VIII = (5 + 1 + 1 + 1) = 8.

(iii) XV = (10 + 5) = 15. (iv) LX = (50 + 10) = 60.

RULE 3 *A smaller numeral written to the left of a larger numeral is always subtracted from the larger numeral.*

CAUTIONS (i) V, L and D are never subtracted.

(ii) I can be subtracted from V and X only.

(iii) X can be subtracted from L and C only.

(iv) C can be subtracted from D and M only.

EXAMPLES (i) IV = (5 − 1) = 4. (ii) IX = (10 − 1) = 9.

(iii) XL = (50 − 10) = 40. (iv) XC = (100 − 10) = 90.

(v) CD = (500 − 100) = 400. (vi) CM = (1000 − 100) = 900.

RULE 4 *When a smaller numeral is placed between two larger numerals, it is always subtracted from the larger numeral immediately following it.*

EXAMPLES (i) XIV = 10 + (5 − 1) = 14. (ii) XIX = 10 + (10 − 1) = 19.

(iii) CXIV = 100 + 10 + (5 − 1) = 114.

Example 2 *Write Roman numeral for each of the numbers from 1 to 20.*

SOLUTION We may write these numbers as given below.

1	2	3	4	5	6	7	8	9	10
I	II	III	IV	V	VI	VII	VIII	IX	X

11	12	13	14	15	16	17	18	19	20
XI	XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX	XX

Example 3 *Express each of the following numbers as a Roman numeral.*

(i) 23 (ii) 26 (iii) 29 (iv) 31 (v) 37 (vi) 39 (vii) 40

(viii) 45 (ix) 49 (x) 51 (xi) 63 (xii) 72 (xiii) 79 (xiv) 84

(xv) 89 (xvi) 90 (xvii) 92 (xviii) 99 (xix) 100

SOLUTION We may write these numbers as given below:

(i) 23 = XXIII (ii) 26 = XXVI (iii) 29 = XXIX

(iv) 31 = XXXI (v) 37 = XXXVII (vi) 39 = XXXIX

(vii) 40 = XL (viii) 45 = XLV (ix) 49 = XLIX

(x) 51 = LI (xi) 63 = LXIII (xii) 72 = LXXII

(xiii) 79 = LXXIX (xiv) 84 = LXXXIV (xv) 89 = LXXXIX

(xvi) 90 = XC (xvii) 92 = XCII (xviii) 99 = XCIX

(xix) 100 = C

Example 4 *Represent the following numbers in the Roman system.*

(i) 137 *(ii)* 198 *(iii)* 302 *(iv)* 389 *(v)* 400 *(vi)* 479 *(vii)* 558
(viii) 596 *(ix)* 715 *(x)* 1222 *(xi)* 2367 *(xii)* 2999

SOLUTION The given numbers may be represented in the Roman system as under.

(i) $137 = 100 + 30 + 7 =$ CXXXVII.

(ii) $198 = 100 + 90 + 8 =$ CXCVIII.

(iii) $302 = 300 + 2 =$ CCCII.

(iv) $389 = 300 + 80 + 9 =$ CCCLXXXIX.

(v) $400 = 500 - 100 =$ CD.

(vi) $479 = 400 + 70 + 9 =$ CDLXXIX.

(vii) $558 = 500 + 50 + 8 =$ DLVIII.

(viii) $596 = 500 + 90 + 6 =$ DXCVI.

(ix) $715 = 700 + 10 + 5 =$ DCCXV.

(x) $1222 = 1000 + 200 + 20 + 2 =$ MCCXXII.

(xi) $2367 = 2000 + 300 + 60 + 7 =$ MMCCCLXVII.

(xii) $2999 = 2000 + 900 + 90 + 9 =$ MMCMXCIX.

Example 5 *Write each of the following in Hindu-Arabic numeral.*

(i) XXIV *(ii)* XLVI *(iii)* LXXXVI *(iv)* XCIX *(v)* CLXVI *(vi)* CCXXVI
(vii) CCCXL *(viii)* CDXLVI *(ix)* MCCLXIX *(x)* M$\overline{\text{V}}$CMXCIX

SOLUTION We have

(i) XXIV $= 20 + (5 - 1) = 24$.

(ii) XLVI $= (50 - 10) + 5 + 1 = 40 + 6 = 46$.

(iii) LXXXVI $= 50 + 30 + 6 = 86$.

(iv) XCIX $= (100 - 10) + (10 - 1) = 90 + 9 = 99$.

(v) CLXVI $= 100 + 50 + 10 + 5 + 1 = 166$.

(vi) CCXXVI $= 200 + 20 + 5 + 1 = 226$.

(vii) CCCXL $= 300 + (50 - 10) = 300 + 40 = 340$.

(viii) CDXLVI $= (500 - 100) + (50 - 10) + 5 + 1 = 400 + 40 + 6 = 446$.

(ix) MCCLXIX $= 1000 + 200 + 50 + 10 + (10 - 1) = 1000 + 200 + 50 + 10 + 9 = 1269$.

(x) M$\overline{\text{V}}$CMXCIX $= (5000 - 1000) + (1000 - 100) + (100 - 10) + (10 - 1)$
$= 4000 + 900 + 90 + 9 = 4999$.

Example 6 *Show that each of the following is meaningless. Give reason in each case.*

(i) XXXX (ii) VX (iii) IC (iv) XVV

SOLUTION (i) No symbol is repeated more than three times.

∴ XXXX is wrong.

(ii) V, L, D are never subtracted.

∴ VX is wrong.

(iii) I can be subtracted from V and X only.

∴ IC is wrong.

(iv) V, L, D are never repeated.

∴ XVV is wrong.

Example 7 *Add the following numerals without converting them to Hindu numerals.*

(i) CCCXXXII + CCCXIII (ii) LXXXVII + LXXVIII

SOLUTION We have

(i)

	D	XL	V
	C C C	X X X	I I
+	C C C	X	I I I
	DC	XL	V or DCXLV.

∴ CCCXXXII + CCCXIII = DCXLV.

(ii)

	C	L	X	V
	L	X X X	V	I I
+	L	X X	V	I I I
	C	L	X	V or CLXV.

∴ LXXXVII + LXXVIII = CLXV.

Multiplication of Roman Numerals

To multiply two Roman numerals without converting them to Hindu numerals, we follow the steps given below.

1. Make a table with 2 columns.
2. Write the two numbers to be multiplied in the first row.
3. Make the next row by halving the first number (discarding remainders) and doubling the second. Continue until there is nothing to halve.
4. Cross out all the rows where the left number is even.
5. Add the remaining numbers in the second column.

The sum so obtained is the product of the two given numbers.

Example 8 *Without converting to Hindu numerals, find the product of the following pairs of landmark numbers.*

(i) V × L (ii) L × D (iii) V × D

SOLUTION We have

(i) V × L

Halving ↓			Doubling ↓
	V	L	
	~~II~~	~~C~~	
	I	CC	
		CCL ← Product	

∴ V × L = CCL.

(ii) $L \times D$

Halving ↓ | Doubling ↓

~~L~~	~~D~~
XXV	M
~~XII~~	~~MM~~
~~VI~~	~~$\overline{IV}$~~
III	$\overline{VIII}$
I	$\overline{XVI}$
	$\overline{XXV}$ ← Product

$\therefore \quad L \times D = \overline{XXV}$.

(iii) $V \times D$

Halving ↓ | Doubling ↓

V	D
~~II~~	~~M~~
I	MM
	MMD ← Product

$\therefore \quad V \times D = MMD$.

EXERCISE 4A

1. Express each of the following as a Roman numeral.

(i) 2 (ii) 8 (iii) 14 (iv) 29
(v) 36 (vi) 43 (vii) 54 (viii) 61
(ix) 73 (x) 81 (xi) 91 (xii) 95
(xiii) 99 (xiv) 105 (xv) 114

2. Express each of the following as a Roman numeral.

(i) 164 (ii) 195 (iii) 226 (iv) 341
(v) 475 (vi) 596 (vii) 611 (viii) 759
(ix) 1084 (x) 1333 (xi) 2987 (xii) 3685

3. Write each of the following as a Hindu-Arabic numeral.

(i) XXVII (ii) XXXIV (iii) XLV (iv) LIV
(v) LXXIV (vi) XCI (vii) XCVI (viii) CXI
(ix) CLIV (x) CCXXIV (xi) CCCLXV (xii) CDXIV
(xiii) CDLXIV (xiv) DVI (xv) DCCLXVI

4. Show that each of the following is meaningless. Give reason in each case.

(i) VC (ii) IL (iii) VVII (iv) IXX

HINT (i) V is never subtracted.
(ii) I can be subtracted from V and X only.
(iii) V, L, D are never repeated.
(iv) IX cannot occur to the left of X.

5. Add the following Roman numerals without converting them to Hindu numerals.

(i) XXVIII + LIX (ii) XCIX + LVIII

(iii) CCLXXXVI + CXLVII (iv) CCCLXIII + CCCXXIX

6. Find the following products.

(i) VII × IX (ii) LVI × XXIV (iii) XXVIII × LXIV (iv) LIV × LXVI

The Idea of a Base

The Egyptian Number System

The Egyptians developed a written number system around 3000 BCE, using hieroglyphic symbols. They used peculiar landmark numbers to group and represent a given number.

The landmark numbers and the hieroglyphic symbols used in the Egyptian number system are as follows.

Landmark numbers	$1\ (=10^0)$	10^1	10^2	10^3	10^4	10^5	10^6	10^7
Hieroglyphic symbols	(line or vertical stroke)	(heel bone)	(coiled rope)	(flower)	(bent finger)	(tadpole)	(god with raised arms)	(sun)

It is clear from the above table that:

(i) the first landmark number is 1 and each next landmark number is 10 times the previous one;

(ii) all the landmark numbers are powers of 10. Such a number system having landmark numbers in which the

(a) first landmark number is 1; and

(b) every next landmark number is obtained by multiplying the current landmark number by some fixed number *n*, is said to be a base-*n* number system.

In the Egyptian number system, every next landmark number is obtained by multiplying the current landmark number by 10. So, it is a base-10 system. A base-10 number system is also called a decimal number system.

Representation of Numbers in Egyptian Number System

The Egyptian number system uses a repetitive and additive method to represent numbers. Thus, to represent a number, the appropriate symbol is repeated as many times as needed.

Thus, 3 is written as ||| (1 + 1 + 1);

30 is written as ∩∩∩ (10 + 10 + 10);

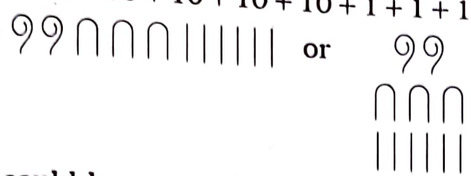
300 is written as 𓍢𓍢𓍢 (100 + 100 + 100), and so on.

So, a given number is counted in groups of the landmark numbers, starting from the largest landmark number less than the given number.

For example,

$236 = 200 + 30 + 6$

$= 100 + 100 + 10 + 10 + 10 + 1 + 1 + 1 + 1 + 1 + 1$

= 

Numerals could be written horizontally in any direction (from left to right or right to left) or vertically from top to bottom.

Since symbols are repeated, the value of a symbol always remains the same irrespective of its place in the numeral. So, the Egyptian number system is not a place-value system.

Example 1 *Represent the following numbers in the Egyptian system.*

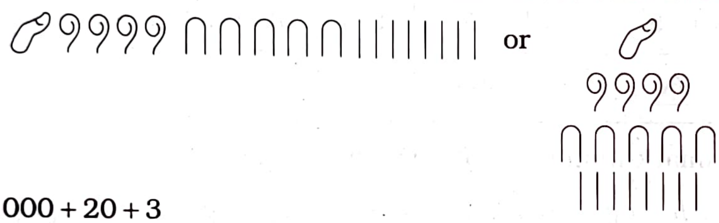
(i) 10458 (ii) 1023 (iii) 2660 (iv) 784

(v) 1111 (vi) 70707

SOLUTION We have

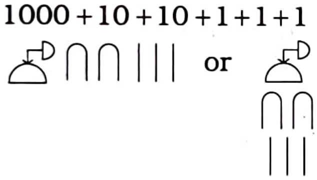
(i) $10458 = 10000 + 400 + 50 + 8$

$= 10000+100+100+100+100+10+10+10+10+10+1+1+1+1+1+1+1+1$

= 

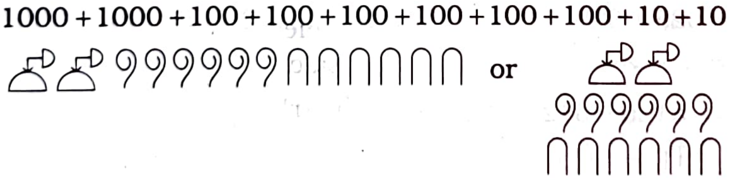
(ii) $1023 = 1000 + 20 + 3$

$= 1000 + 10 + 10 + 1 + 1 + 1$

= 

(iii) $2660 = 2000 + 600 + 60$

$= 1000 + 1000 + 100 + 100 + 100 + 100 + 100 + 100 + 10 + 10 + 10 + 10 + 10 + 10$

= 

(iv) $784 = 700 + 80 + 4$

$= 100 + 100 + 100 + 100 + 100 + 100 + 100 + 10 + 10 + 10 + 10 + 10 + 10 + 10 + 10$
$+1+1+1+1$

= 

(v) $1111 = 1000 + 100 + 10 + 1 =$ 

(vi) $70707 = 70000 + 700 + 7$

=